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Quadrature Formulas of Quasi-Interpolation Type for Singular Integrals with Hilbert Kernel

Du Jinyuan

Department of Mathematics, Wuhan University, Wuhan, Hubei, 430072, People's Republic of China

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In this paper, we first establish quadrature formulas of trigonometric interpolation type for proper integrals of periodic functions with periodic weight, then we use the method of separation of singularities to derive those for corresponding singular integrals with Hilbert kernel. The trigonometric precision, the type, the estimate of the remainder, and the convergence of each quadrature formula derived here are also established. © 1998 Academic Press

1. INTRODUCTION

In [1], the author systematically discussed singular quadrature formulas with the highest trigonometric precision for the singular integral with Hilbert kernel

$$(\mathbf{H}f)(t)=\int_0^{2\pi} w(\tau)\, f(\tau) \cot \frac{\tau - t}{2}\, d\tau, \qquad t \in [0, 2\pi), \tag{1.1}$$

where $w(\tau)$ is a given non-negative function with period 2π which is known as the weight function, $f(\tau)$ is a function with period 2π, and the integral is understood as the Cauchy principal value integral with $(\mathbf{H}f)(0) = \lim_{\delta \to 0} \int_\delta^{2\pi-\delta} w(\tau)\, f(\tau) \cot \frac{1}{2}\tau\, d\tau$. For its existence $w(\tau)$ and $f(\tau)$ are assumed further to be Hölder continuous, denoted as $w, f \in H_{2\pi}$.

In the present paper, we shall consider a kind of quadrature formulas for (1.1) regarded as of quasi-interpolation type. The quadrature formulas with the highest trigonometric precision established in [1] are also of this type, the nodes of which must be chosen appropriately. Therefore they are not very convenient in applications. The nodes of a quadrature formula of this type may be chosen arbitrarily, so it is sometimes more convenient in applications.

We first establish some results for trigonometric interpolation, then use them to establish quadrature formulas of trigonometric interpolation type

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of proper integrals for periodic functions with periodic weight functions. Finally we use the method of separation of singularities to derive quadrature formulas for (1.1). The trigonometric precision, the type, estimate of the remainder, and the convergence of each quadrature formula derived here are established. The application of these results in this paper to methods for the numerical solution of singular integral equations with Hilbert kernel will be given in a forthcoming paper.

2. TRIGONOMETRIC INTERPOLATION

Let H_n^T denote the class of all trigonometric polynomials of degree not greater than n. We regard $H_n^T = \{0\}$ if $n < 0$.

Let $H_n^T(\alpha)$ denote the family of trigonometric polynomials of the form

$$a_n \sin(nt + \alpha) + T_{n-1}(t), \qquad T_{n-1} \in H_{n-1}^T, \quad 0 \leqslant \alpha < \pi, \tag{2.1}$$

in which a_n is called the coefficient of the term of degree n, and regard $H_n^T(\alpha) = \{0\}$ if $n < 0$. It is obvious that any trigonometric polynomial of degree n ($n > 0$) cannot belong to two different classes $H_n^T(\alpha_1)$ and $H_n^T(\alpha_2)$ ($\alpha_1 \neq \alpha_2$), while $H_0^T(\alpha)$ is $\{0\}$ when $\alpha = 0$ and the set of all constants when $\alpha \neq 0$.

For n different points $t_1, t_2, ..., t_n$ in $[0, 2\pi)$, we set

$$\Delta_n(\tau) = \prod_{j=1}^{n} \sin \frac{\tau - t_j}{2}. \tag{2.2}$$

For $f \in H_{2\pi}$ we introduce the trigonometric interpolation operator (TIO) of the form

$$(\mathbf{T}_n^{\Delta} f)(\tau) = \sum_{j=1}^{n} T_{n,j}^{\Delta}(\tau)\, f(t_j), \tag{2.3}$$

where

$$T_{n,j}^{\Delta}(\tau) = \begin{cases} \dfrac{\Delta_n(\tau)}{2\Delta_n'(t_j)} \csc \dfrac{\tau - t_j}{2}, & \text{if } n \text{ is odd,} \\[2ex] \dfrac{\Delta_n(\tau) \sin((\tau - t_j)/2 + \alpha - \phi)}{2\Delta_n'(t_j) \sin(\alpha - \phi)} \csc \dfrac{\tau - t_j}{2}, & \text{if } n \text{ is even,} \end{cases} \tag{2.4}$$

$$\phi = \left[\frac{\pi}{2} - \frac{1}{2} \sum_{r=1}^{n} t_r \right]_{\pi}, \tag{2.5}$$

$$0 \leqslant \alpha < \pi, \qquad \alpha \neq \phi, \tag{2.6}$$

and $[\theta]_\pi$ denotes the number congruent to θ (mod π) in $[0, \pi)$.

Remark 2.1. When $n=2m$, $\Delta_n(\tau)\in H_m^T(\phi)$, we call ϕ the characteristic number of $\Delta_n(\tau)$.

Obviously,

$$T_{n,j}^{\Delta}\in\begin{cases}H_{m-1}^T, & \text{if} \quad n=2m-1,\\ H_m^T(\alpha), & \text{if} \quad n=2m,\end{cases} \tag{2.7}$$

therefore

$$\mathbf{T}_n^{\Delta}\colon H_{2\pi}\to\begin{cases}H_{m-1}^T, & \text{if} \quad n=2m-1,\\ H_m^T(\alpha), & \text{if} \quad n=2m.\end{cases} \tag{2.8}$$

Here $\mathbf{T}_{2m-1}^{\Delta}$ is just the known classical trigonometric interpolation by using an odd number of knots; $\mathbf{T}_{2m}^{\Delta}$ is called the trigonometric interpolation in $H_m^T(\alpha)$ by using an even number of knots [2,3].

Let

$$\delta_n^{\Delta}=\mathbf{I}-\mathbf{T}_n^{\Delta}, \tag{2.9}$$

where $\mathbf{I}$ is the identity operator. Then δ_n^{Δ} is called the remainder of TIO (2.3). By the uniqueness of the classical trigonometric interpolation and of the trigonometric interpolation in $H_m^T(\alpha)$ (cf. Lemma 2.1 in [2]), we have

LEMMA 2.1.

$$\ker\{\delta_n^{\Delta}\}=\begin{cases}H_{m-1}^T, & \textit{if} \quad n=2m-1,\\ H_m^T(\alpha), & \textit{if} \quad n=2m.\end{cases}$$

EXAMPLE 2.1. If we take $\alpha=[\pi/2+\phi]_\pi$, then

$$(\mathbf{T}_n^{\Delta}f)(\tau)=\begin{cases}\displaystyle\sum_{j=1}^{n}\frac{\Delta_n(\tau)}{2\Delta_n'(t_j)}\csc\frac{\tau-t_j}{2}f(t_j), & \text{if } n \text{ is odd},\\ \displaystyle\sum_{j=1}^{n}\frac{\Delta_n(\tau)}{2\Delta_n'(t_j)}\cot\frac{\tau-t_j}{2}f(t_j), & \text{if } n \text{ if even},\end{cases} \tag{2.10}$$

$$\ker\{\delta_n^{\Delta}\}=\begin{cases}H_{m-1}^T, & \text{if} \quad n=2m-1,\\ H_m^T([\pi/2+\phi]_\pi), & \text{if} \quad n=2m.\end{cases} \tag{2.11}$$

Such a trigonometric interpolation operator is said to be of normal form. It is just the proximal interpolant given in [4] by Schoenberg.

When f possesses analyticity, we may give $\delta_n^{\Delta}f$ a clear representation. Assume that f is a 2π-periodic function analytic on the rectangular domain $D_r=\{z, 0\leqslant \operatorname{Re} z\leqslant 2\pi, |\operatorname{Im} z|\leqslant r\}$ $(r>0)$ with the boundary ∂D_r. We denote $f\in AP(D_r)$ and discuss the remain of its TIO in the following.

In the first place, we assume $f \in C'_{2\pi}$. Let

$$F(\tau, t) = \begin{cases} [f(\tau) - f(t)] \cot \dfrac{\tau - t}{2}, & \text{if} \quad \tau \not\equiv t \pmod{2\pi}, \\ 2f'(t), & \text{if} \quad \tau \equiv t \pmod{2\pi}. \end{cases} \tag{2.12}$$

It is obvious that $F(\tau, t)$ is a continuous function of τ and t with period 2π. Sometimes we treat t as a parameter and write $F(\tau, t)$ as $F_t(\tau)$.

LEMMA 2.2. *If $f \in H_n^T(\beta)$, then $F_t(\tau) \in H_n^T([\frac{1}{2}\pi + \beta]_\pi)$, or, more precisely,*

$$F(\tau, t) = a_n[\cos(n\tau + \beta) + \cos(nt + \beta)] + \sum_{j=1}^{n-1} [A_{n-j}(t) \sin j\tau + B_{n-j}(t) \cos j\tau], \tag{2.13}$$

where a_n is the coefficient of the term of degree n of f, A_j and $B_j \in H_j^T$.

Proof. It is sufficient to prove the case $f(\tau) = \sin(n\tau + \beta)$. This is easy. In fact, denoting $w = e^{i\tau}$ and $z = e^{it}$, we have

$$\begin{aligned} &[\sin(n\tau + \beta) - \sin(nt + \beta)] \cot \frac{\tau - t}{2} \\ &\quad = \frac{1}{2} [e^{i\beta}(w^n - z^n) + e^{-i\beta}(w^{-n} - z^{-n})] \frac{w + z}{w - z} \\ &\quad = \frac{1}{2} e^{i\beta} \left[w^n + 2 \sum_{j=1}^{n-1} w^{n-j} z^j + z^n \right] \\ &\qquad + \frac{1}{2} e^{-i\beta} \left[w^{-n} + 2 \sum_{j=1}^{n-1} w^{-n+j} z^{-j} + z^{-n} \right] \\ &\quad = \cos(n\tau + \beta) + \cos(nt + \beta) \\ &\qquad - 2 \sum_{j=1}^{n-1} [\sin((n-j)\, t + \beta) \sin j\tau - \cos((n-j)\, t + \beta) \cos j\tau]. \end{aligned}$$

Let

$$\Delta_n(\tau, t) = \begin{cases} \left[\Delta_n(\tau) - \Delta_n(t) \cos \dfrac{\tau - t}{2} \right] \csc \dfrac{\tau - t}{2}, & \text{if} \quad n = 2m - 1, \\ [\Delta_n(\tau) - \Delta_n(t)] \cot \dfrac{\tau - t}{2}, & \text{if} \quad n = 2m. \end{cases} \tag{2.14}$$

LEMMA 2.3. *For any fixed t,*

$$\Lambda_n(\cdot, t) \in \begin{cases} H^T_{m-1}, & \text{if} \quad n = 2m-1, \\ H^T_m(\alpha), & \text{if} \quad n = 2m, \end{cases} \tag{2.15}$$

where α is given in Example 2.1.

Proof. The second conclusion follows immediately from (2.13) and Remark 2.1. For the case $n = 2m-1$, let $p_{m-1}(\tau) = \prod_{j=2}^{n} \sin \frac{1}{2}(\tau - t_j)$ with $p_o = 1$, the conclusion results from (2.13) and

$$\Lambda_n(\tau, t) = \sin \frac{t - t_1}{2} \left[p_{m-1}(\tau) - p_{m-1}(t) \right] \cot \frac{\tau - 1}{2} + \cos \frac{t - t_1}{2} \, p_{m-1}(\tau). \tag{2.16}$$

THEOREM 2.1. *If $f \in AP(D_r)$, then*

$$(\mathbf{T}^{\Delta}_n f)(\tau) = \frac{1}{4\pi i} \int_{\partial D_r} f(z) \frac{\Lambda_n(\tau, z)}{\Delta_n(z)} \, dz, \tag{2.17}$$

$$(\delta^{\Delta}_n f)(\tau) = \begin{cases} \dfrac{1}{4\pi} \displaystyle\int_{\partial D_r} \frac{\Delta_n(\tau)}{\Delta_n(z)} f(z) \csc \frac{z - \tau}{2} \, dz, & \text{if} \quad n = 2m-1, \\ \dfrac{1}{4\pi i} \displaystyle\int_{\partial D_r} \frac{\Delta_n(\tau)}{\Delta_n(z)} f(z) \cot \frac{z - \tau}{2} \, dz, & \text{if} \quad n = 2m, \end{cases} \tag{2.18}$$

or

$$(\delta^{\Delta}_n f)(\tau) = \begin{cases} \dfrac{1}{2\pi} \operatorname{Re} \left\{ i \displaystyle\int_{ir}^{2\pi + ir} \frac{\Delta_n(\tau)}{\Delta_n(z)} f(z) \csc \frac{z - \tau}{2} \, dz \right\}, \\ \qquad \text{if} \quad n = 2m-1, \\ \dfrac{1}{2\pi} \operatorname{Re} \left\{ i \displaystyle\int_{ir}^{2\pi + ir} \frac{\Delta_n(\tau)}{\Delta_n(z)} f(z) \cot \frac{z - \tau}{2} \, dz \right\}, \\ \qquad \text{if} \quad n = 2m, \end{cases} \tag{2.19}$$

and

$$\|\delta^{\Delta}_n f\| \leqslant \coth\left(\frac{r}{2}\right) \|f\|_r \, \|(\Delta_n)^{-1}\|_r, \tag{2.20}$$

where $\mathbf{T}^{\Delta}_n$ is a TIO of normal form and $\|\cdot\|$ and $\|\cdot\|_r$ denote the sup-norms of a function on $[0, 2\pi]$ and the line-segment $z = x + ir$ $(0 \leqslant x \leqslant 2\pi)$, respectively.

Proof. We prove only the case $n=2m-1$; the case $n=2n$ is similar. By (2.16) and (2.13)

$$T_{m-1}(\tau)=\frac{1}{4\pi i}\int_{\partial D_r} f(z)\,\frac{\Delta_n(\tau, z)}{\Delta_n(z)}\,dz\in H^T_{m-1}. \tag{2.21}$$

In (2.21), if $\Delta_n(0)=\Delta_n(2\pi)=0$, we understand it as the Cauchy principal value integral. Again by the residue theorem, for τ in the interior of ∂D_r

$$f(\tau)=\frac{1}{4\pi i}\int_{\partial D_r} f(z)\cot\frac{z-\tau}{2}\,dz, \tag{2.22}$$

and this still holds for $\tau=0$ (2π) by the extended residue theorem due to Jian-ke Lu [5, p. 75]. In fact,

$$\frac{1}{4\pi i}\int_{\partial D_r} f(z)\cot\frac{z}{2}\,dz=[\operatorname{sp}(0)\operatorname{res}(0)+\operatorname{sp}(2\pi)\operatorname{res}(2\pi)]=f(0),$$

where $\operatorname{sp}(x)$ denotes the span at x with respect to ∂D_r, for example, $\operatorname{sp}(0)=\operatorname{sp}(2\pi)=\frac{1}{2}$, and $\operatorname{res}(x)$ the residue of the integrand $f(z)\cot z/2$ at x. Thus,

$$f(\tau)-T_{m-1}(\tau)=\frac{1}{4\pi i}\int_{\partial D_r}\frac{\Delta_n(\tau)}{\Delta_n(z)}\,f(z)\csc\frac{z-\tau}{2}\,dz, \tag{2.23}$$

in particular,

$$T_{m-1}(t_j)=f(t_j)=(\mathbf{T}^{\Delta}_n f)(t_j) \qquad (j=1, ..., n).$$

Finally, by Lemma 2.1, we obtain (2.17) and (2.18). Noting that f possesses the periodicity and the Schwarz symmetry (i.e., $f(\bar{z})=\overline{f(z)}$) by $f(\mathscr{R})\subseteq\mathscr{R}$ ($\mathscr{R}$ denotes the set of real numbers), and the principle of the Schwarz symmetric extension, so does the integrand in (2.23). We obtain (2.19) from (2.18); (2.20) follows from (2.19).

Remark 2.2. For further use, we need to rewrite (2.18) as

$$(\delta^{\Delta}_n f)(\tau)=\begin{cases}\dfrac{\Delta_n(\tau)}{4\pi i}\displaystyle\int_{\partial D_r}\frac{f(z)}{\Delta_n(z)}\csc\frac{z-\tau}{2}\,dz, & \text{if}\quad n=2m-1,\\[2ex] \dfrac{\Delta_n(\tau)}{4\pi i}\displaystyle\int_{\partial D_r}\frac{f(z)}{\Delta_n(z)}\cot\frac{z-\tau}{2}\,dz, & \text{if}\quad n=2m.\end{cases} \tag{2.24}$$

To do this, we understand the integrals appearing in the right-hand side of (2.24) as the singular integrals when either $\Delta_n(0)=0$ or $\tau=0$. In particular, they are of higher order when both $\Delta_n(0)=0$ and $\tau=0$ [5, 6].

Remark 2.3. The method applied here is also valid for the general TIO given in (2.3).

EXAMPLE 2.2. Taking $\Delta_n(\tau)=\sin(\frac{1}{2}n\tau+\theta)$ with an arbitrary real number θ, from (2.20) we obtain

$$\|\delta_n^{\Delta} f\| \leqslant \coth\left(\tfrac{1}{2}r\right) \sinh^{-1}\left(\tfrac{1}{2}nr\right) \|f\|_r = O(e^{-(1/2)\,nr}) \qquad \text{as} \quad n \to \infty. \tag{2.25}$$

Analogously, for the more general case, we have the following.

COROLLARY 2.1. *If $f \in AP(D_r)$ then*

$$\|\delta_n^{\Delta} f\| \leqslant \coth\left(\tfrac{1}{2}r\right) \|f\|_r \sinh^{-n}\left(\tfrac{1}{2}r\right). \tag{2.26}$$

In particular, if $r > 2\ln(1+\sqrt{2})$ then $\|\delta_n^{\Delta} f\| \to 0$ as $n \to \infty$.

3. QUADRATURE FORMULAS FOR PROPER INTEGRALS

In this section, we discuss the quadrature formulas for the proper integral

$$\mathbf{D}f = \int_0^{2\pi} w(\tau)\, f(\tau)\, d\tau. \tag{3.1}$$

It is well known that the general form of the quadrature formula for (3.1) with the nodes $\{t_1, ..., t_n\}$ is

$$\int_0^{2\pi} w(\tau)\, f(\tau)\, d\tau \approx \sum_{j=1}^{n} H_j f(t_j) =: \mathbf{Q}_n^{\Delta D} f. \tag{3.2}$$

The right-hand side is known as the quadrature sum.

We may arbitrarily choose the nodes $\{t_j\}$ and the coefficients $\{H_j\}$, in as much as the structure of the quadrature formula is arbitrary. Certainly, we hope that (3.2) will possess suitable accuracy for the class of trigonometric polynomials through the appropriate choice of $\{t_j\}$ and $\{H_j\}$.

Let

$$\mathbf{R}_n^{\Delta D} f = \mathbf{D}f - \mathbf{Q}_n^{\Delta D} f \tag{3.3}$$

which is called the remainder of (3.2).

DEFINITION 3.1. If $H_{m-1}^T \subseteq \ker\{\mathbf{R}_n^{\Delta D}\}$, but there exists some $T_m \in H_m^T$ such that $\mathbf{R}_n^{\Delta D} T_m \neq 0$, then we define the quadrature formula (3.2) to have the trigonometric precision of order $m-1$, denoted as $\mathrm{pr}(\mathbf{Q}_n^{\Delta D}) = m-1$.

We mention that (3.2) does not possess the trigonometric precision of order n, however we choose $\{t_j\}$ and $\{H_j\}$. This may be easily seen by considering the trigonometric polynomial Δ_n^2. But from [1], we know that there exists the quadrature formula with the trigonometric precision of order $\operatorname{pr}(\mathbf{Q}_n^{\Delta D}) = n-1$. For such a quadrature formula with the highest trigonometric precision, the nodes must be chosen suitably. We may both arbitrarily choose the nodes and ensure that the quadrature formula obtained possesses a certain trigonometric precision.

DEFINITION 3.2. If $\operatorname{pr}(\mathbf{Q}_n^{\Delta D}) \geqslant [\frac{1}{2}(n-1)]$, we say (3.2) is the quadrature formula of interpolation type, where $|x|$ denotes the integral part of x.

We set

$$\mathbf{Q}_n^{\Delta D} f = \mathbf{D}\mathbf{T}_n^{\Delta} f, \tag{3.4}$$

where $\mathbf{T}_n^{\Delta}$ is given by (2.3). Then (3.2) is the quadrature formula of trigonometric interpolation type by Lemma 2.1 and

$$H_j = \begin{cases} \dfrac{1}{2\Delta_n'(t_j)} \displaystyle\int_0^{2\pi} w(\tau)\, \Delta_n(\tau) \csc \frac{\tau - t_j}{2}\, d\tau, \\ \qquad \text{if} \quad n = 2m-1, \\ \dfrac{1}{2\Delta_n'(t_j)} \displaystyle\int_0^{2\pi} w(\tau)\, \Delta_n(\tau)\, \frac{\sin[(1/2)(\tau - t_j) + \alpha - \phi]}{\sin(\alpha - \phi)} \csc \frac{\tau - t_j}{2}\, d\tau, \\ \qquad \text{if} \quad n = 2m. \end{cases} \tag{3.5}$$

On the other hand, for the case $n = 2m-1$ (odd), if (3.2) is of trigonometric interpolation type, then for $T_{n,j}^{\Delta}$ in (2.4), $\mathbf{D}(T_{n,j}^{\Delta}) = \mathbf{Q}_n^{\Delta D}(T_{n,j}^{\Delta}) = H_j$, i.e., H_j is given by (3.5). Therefore the quadrature formula of trigonometric interpolation type is unique for the fixed nodes $\{t_j\}$.

THEOREM 3.1. *In case $n = 2m-1$ (odd), the quadrature formula* (3.2) *is of trigonometric interpolation type iff its coefficients*

$$H_j = \frac{1}{2\Delta_n'(t_j)} \int_0^{2\pi} w(\tau)\, \Delta_n(\tau) \csc \frac{\tau - t_j}{2}\, d\tau \qquad (j = 1, ..., n). \tag{3.6}$$

In case $n = 2m$ (even), for the fixed set of notes $\{t_j\}$, the quadrature formula of trigonometric interpolation type is not unique in general. To see this, we must introduce the concept of the type of quadrature formula (3.2).

It is interesting to note that there is an essential difference between the concepts of algebraic precision and of trigonometric precision. A quadrature formula having the algebraic precision of order $m-1$ is not exact for any algebraic polynomial of degree m, but a quadrature formula having the trigonometric precision of order $m-1$ is exact for certain trigonometric polynomials of degree m and not so for the rest. This will be verified as follows.

DEFINITION 3.3. If $\operatorname{pr}(\mathbf{Q}_n^{\Delta D})=m-1$ and $H_m^T(\alpha)\subseteq\ker(\mathbf{R}_n^{\Delta D})$, we say (3.2) is of $H_m^T(\alpha)$ type, denoted as $\operatorname{ty}\{\mathbf{Q}_n^{\Delta D}\}=H_m^T(\alpha)$.

Obviously, (3.2) could not possess two different types. In fact, if $\operatorname{ty}\{\mathbf{Q}_n^{\Delta D}\}=H_m^T(\alpha)$ and $\operatorname{ty}\{\mathbf{Q}_n^{\Delta D}\}=H_m^T(\beta)$ with $\alpha\neq\beta$, then both $\sin m\tau$ and $\cos m\tau$ belong to $\ker\{\mathbf{R}_n^{\Delta D}\}$. Thus $\operatorname{pr}(\mathbf{Q}_n^{\Delta D})\geqslant m$, which is a contradiction. Moreover, any quadrature formula possesses some type. If $\operatorname{pr}(\mathbf{Q}_n^{\Delta D})=m-1$, setting $f(\tau)=\cos m\tau$ and $g(\tau)=\sin m\tau$, we know either $c_m=\mathbf{R}_n^{\Delta D}(f)\neq 0$ or $s_m=\mathbf{R}_n^{\Delta D}(g)\neq 0$. Taking

$$\alpha=\begin{cases}\operatorname{arc\,ctg}\left(-\dfrac{c_m}{s_m}\right), & \text{if } s_m\neq 0,\\[2ex] 0, & \text{if } s_m=0,\end{cases}\tag{3.7}$$

and $\psi(\tau)=\sin(m\tau+\alpha)$, we have $\mathbf{R}_n^{\Delta D}(\psi)=0$.

DEFINITION 3.4. If $\operatorname{ty}\{\mathbf{Q}_n^{\Delta D}\}\supseteq H_r^T(\alpha)$ where $r=[\frac{1}{2}(n+1)]$, we then define (3.2) to be of α-interpolation type.

Obviously, a quadrature formula of α-interpolation type is surely of interpolation type. For $n=2m-1$ (odd), by Theorem 3.1 there exists only a quadrature formula of α-interpolation type and α is just that given in (3.7). Therefore the trigonometric precision of a quadrature formula may completely portray the character of its interpolation. But for $n=2m$ (even) there are many different quadrature formulas of interpolation type, more specifically, there is one quadrature formula of α-interpolation type at least for each α $(\neq\phi)$, and so the character of a quadrature formula of interpolation type must be portrayed via its type.

THEOREM 3.2. *In case $n=2m$ (even) and α is as* (2.6), *the quadrature formula* (3.2) *is of α-interpolation type iff its coefficients satisfy*

$$H_j=\frac{1}{2\Delta_n'(t_j)}\int_0^{2\pi}w(\tau)\,\Delta_n(\tau)\,\frac{\sin[(1/2)(\tau-t_j)+\alpha-\phi]}{\sin(\alpha-\phi)}\times\csc\frac{\tau-t_j}{2}\,d\tau\qquad(j=1,...,n).\tag{3.8}$$

Proof. By (2.7), we have (3.8) $\Leftrightarrow H_j=\mathbf{D}(T_{n,j}^{\Delta})\Leftrightarrow\mathbf{Q}_n^{\Delta D}f=\mathbf{D}\mathbf{T}_n^{\Delta}f\Leftrightarrow$ $H_n^T(\alpha)\subseteq\operatorname{ty}\{\mathbf{Q}_n^{\Delta D}\}$.

Remark 3.1. It is possible that a quadrature formula of α-interpolation type is possibly the same as a quadrature formula of β-interpolation type though $\alpha \neq \beta$, for example, the quadrature formula with the highest trigonometric precision established in [1]. But in this instance the set of nodes $\{t_j\}$ is not arbitrary.

EXAMPLE 3.1. The quadrature formula (3.2) is of trigonometric interpolation type when $n = 2m - 1$ or of α-interpolation type when $n = 2m$, where α is given in Example 2.1, iff its coefficients are

$$H_j = \begin{cases} \dfrac{1}{2\Delta_n'(t_j)} \displaystyle\int_0^{2\pi} w(\tau)\, \Delta_n(\tau) \csc \frac{\tau - t_j}{2}\, d\tau, & \text{if} \quad n = 2m-1, \\ \dfrac{1}{2\Delta_n'(t_j)} \displaystyle\int_0^{2\pi} w(\tau)\, \Delta_n(\tau) \cot \frac{\tau - t_j}{2}\, d\tau, & \text{if} \quad n = 2m, \end{cases} \tag{3.9}$$

and the remainder is

$$\mathbf{R}_n^{\Delta D} f = \mathbf{D} \delta_n^{\Delta} f, \tag{3.10}$$

where δ_n^{Δ} is as (2.9).

Such a quadrature formula of trigonometric interpolation type is said to be of normal form. From now on, we only discuss the quadrature formulas of normal form although the method may be applied to general cases. First we extend (1.1) to the whole complex plane $\mathscr{C}$,

$$(\mathbf{H} f)(z) = \int_0^{2\pi} w(\tau)\, f(\tau) \cot \frac{\tau - z}{2}\, d\tau, \qquad z \in \mathscr{C}, \tag{3.11}$$

and introduce another operator with the cosecant kernel for the function in the class $\tilde{H}_{2\pi}$ (the class of the Hölder continuous functions with $f(\tau + 2\pi) = -f(\tau)$):

$$(\tilde{\mathbf{H}} f)(z) = \int_0^{2\pi} w(\tau)\, f(\tau) \csc \frac{\tau - z}{2}\, d\tau, \qquad z \in \mathscr{C}. \tag{3.12}$$

When z is a real number, we understand (3.12) as the Cauchy principal value integral, in particular, for $j = \pm 1, \ldots$, $(\tilde{\mathbf{H}} f)(2j\pi) = (-1)^j (\tilde{\mathbf{H}} f)(0)$ with

$$\begin{aligned} (\tilde{\mathbf{H}} f)(0) &:= \lim_{\delta \to 0} \int_\delta^{2\pi - \delta} w(\tau)\, f(\tau) \csc \frac{\tau}{2}\, d\tau \\ &= \int_0^{2\pi} w(\tau)\, f(\tau) \sin \frac{\tau}{2}\, d\tau + \int_0^{2\pi} w(\tau)\, f(\tau) \cos \frac{\tau}{2} \cot \frac{\tau}{2}\, d\tau. \end{aligned} \tag{3.13}$$

We point out that both $(\mathbf{H}f)(z)$ and $(\tilde{\mathbf{H}}f)(z)$ are not continuous at $z=\tau$ $(\tau \in \mathscr{R})$ except when $w(\tau)\, f(\tau)=0$. In fact, introduce the functions

$$(\mathbf{H}^{+}f)(z)=\begin{cases}(\mathbf{H}f)(z) & \text{if} \quad \operatorname{Im} z>0,\\ 2\pi i w(t)\, f(t)+(\mathbf{H}f)(t) & \text{if} \quad z=t,\ \operatorname{Im} z=0;\end{cases} \tag{3.14}$$

$$(\mathbf{H}^{-}f)(z)=\begin{cases}(\mathbf{H}f)(z) & \text{if} \quad \operatorname{Im} z<0,\\ -2\pi i w(t)\, f(t)+(\mathbf{H}f)(t) & \text{if} \quad z=t,\ \operatorname{Im} z=0;\end{cases} \tag{3.15}$$

$$(\tilde{\mathbf{H}}^{+}f)(z)=\begin{cases}(\tilde{\mathbf{H}}f)(z) & \text{if} \quad \operatorname{Im} z>0,\\ 2\pi i w(t)\, f(t)+(\tilde{\mathbf{H}}f)(t) & \text{if} \quad z=t,\ \operatorname{Im} z=0;\end{cases} \tag{3.16}$$

$$(\tilde{\mathbf{H}}^{-}f)(z)=\begin{cases}(\tilde{\mathbf{H}}f)(z) & \text{if} \quad \operatorname{Im} z<0,\\ -2\pi i w(t)\, f(t)+(\tilde{\mathbf{H}}f)(t) & \text{if} \quad z=t,\ \operatorname{Im} z=0;\end{cases} \tag{3.17}$$

then, by the Plemelj–Privalov theorem [5, 7], we know that $\mathbf{H}^{+}f$ and $\mathbf{H}^{-}f$ are, respectively, analytic in $\mathscr{C}^{+}=\{z, \operatorname{Im} z>0\}$ and $\mathscr{C}^{-}=\{z, \operatorname{Im} z<0\}$, as well as in the class $H_{2\pi}$ on $\overline{\mathscr{C}^{+}}=\{z, \operatorname{Im} z\geqslant 0\}$ and $\overline{\mathscr{C}^{-}}=\{z, \operatorname{Im} z\leqslant 0\}$. By analogy, $\tilde{\mathbf{H}}^{+}f$ and $\tilde{\mathbf{H}}^{-}f$ are, respectively, analytic in $\mathscr{C}^{+}$ and $\mathscr{C}^{-}$, as well as in the class $\tilde{H}_{2\pi}$ on $\overline{\mathscr{C}^{+}}$ and $\overline{\mathscr{C}^{-}}$. We write them as $\mathbf{H}^{+}f\in A^{+}H_{2\pi}$, $\mathbf{H}^{+}f\in A^{-}H_{2\pi}$, $\tilde{\mathbf{H}}^{+}f\in A^{+}\tilde{H}_{2\pi}$, and $\tilde{\mathbf{H}}^{-}f\in A^{-}\tilde{H}_{2\pi}$, respectively.

Remark 3.2. From the above discussion we know that $\mathbf{H}^{+}\colon H_{2\pi}\to A^{+}H_{2\pi}$, $\mathbf{H}^{-}\colon H_{2\pi}\to A^{-}H_{2\pi}$, $\tilde{\mathbf{H}}^{+}\colon \tilde{H}_{2\pi}\to A^{+}\tilde{H}_{2\pi}$, and $\tilde{\mathbf{H}}\colon \tilde{H}_{2\pi}\to A^{-}\tilde{H}_{2\pi}$.

Let

$$\Delta_n^{*}(z)=\begin{cases}(\tilde{\mathbf{H}}\Delta_n)(z), & \text{if} \quad n=2m-1,\\ (\mathbf{H}\Delta_n)(z), & \text{if} \quad n=2m.\end{cases} \tag{3.18}$$

This function is called the associated function of Δ_n with respect to the weight $w(t)$. It plays an important role in the following discussions. Now we may rewrite the coefficients, given by (3.9), of the quadrature formula of normal form as

$$H_j=\frac{\Delta_n^{*}(t_j)}{2\Delta_n'(t_j)} \qquad (j=1, ..., n). \tag{3.19}$$

In addition, let

$$(\Delta_n^{*})^{\pm}(z)=\begin{cases}(\tilde{\mathbf{H}}^{\pm}\Delta_n)(z), & \text{if} \quad n=2m-1,\\ (\mathbf{H}^{\pm}\Delta_n)(z), & \text{if} \quad n=2m.\end{cases} \tag{3.20}$$

Remark 3.3. $(\Delta_n^{*})^{+}(t_j)=(\Delta_n^{*})^{-}(t_j)=\Delta_n^{*}(t_j)$ by (3.14)–(3.18), therefore, by Remark 3.2, Δ_n^{*} is Hölder continuous at $z=t_j$ and on each of the straight lines $z=t_j+iy$ $(-\infty<y<+\infty)$ and $z=x+ic$ $(-\infty<x<+\infty$, c is an arbitrary real constant).

THEOREM 3.3. *If $f \in AP(D_r)$, then*

$$\mathbf{R}_n^{\Delta D} f = -\frac{1}{4\pi i} \int_{\partial D_r} f(z) \frac{\Delta_n^*(z)}{\Delta_n(z)}\, dz, \tag{3.21}$$

or

$$\mathbf{R}_n^{\Delta D} f = -\frac{1}{2\pi} \operatorname{Re} \left\{ i \int_{ir}^{2\pi + ir} f(z) \frac{\Delta_n^*(z)}{\Delta_n(z)}\, dz \right\} \tag{3.22}$$

and

$$|\mathbf{R}_n^{\Delta D} f| \leqslant \left\| \frac{\Delta_n^*}{\Delta_n} \right\|_r \|f\|_r. \tag{3.23}$$

Proof. We only prove the case $n = 2m-1$. Noting Remark 2.2 and the periodicity of the integrand, we have that

$$\begin{aligned} &\frac{1}{4\pi i} \int_{\partial D_r} \frac{f(z)}{\Delta_n(z)} \csc \frac{z-\tau}{2}\, dz \\ &\quad = \frac{1}{4\pi i} \left\{ \int_{-ir}^{2\pi - ir} - \int_{ir}^{2\pi + ir} \right\} \frac{f(z)}{\Delta_n(z)} \csc \frac{z-\tau}{2}\, dz, \end{aligned} \tag{3.24}$$

hence, by (3.10), (2.24), (3.24), and (3.18), we get

$$\begin{aligned} \mathbf{R}_n^{\Delta D} f &= \frac{1}{4\pi i} \left\{ \int_{ir}^{2\pi + ir} - \int_{-ir}^{2\pi - ir} \right\} f(z) \frac{\Delta_n^*(z)}{\Delta_n(z)}\, dz \\ &= -\frac{1}{4\pi i} \int_{\partial D_r} f(z) \frac{\Delta_n^*(z)}{\Delta_n(z)}\, dz. \end{aligned}$$

Of course, in the case $\Delta_n(0) = 0$, the last integral above should be understood as the Cauchy principal value integral which actually exists because Δ_n^* is Hölder continuous on ∂D_r from Remark 3.3. The rest of the proof is obvious.

COROLLARY 3.1. *When*

$$f \in \begin{cases} H_{m-1}^T, & \text{if } \; n = 2m-1, \\ H_m^T(\alpha), & \text{if } \; n = 2m, \end{cases}$$

where α is that given in Example 2.1, then

$$\frac{1}{4\pi i} \int_{\partial D_r} f(z) \frac{\Delta_n^*(z)}{\Delta_n(z)}\, dz = 0,$$

in particular,

$$\frac{1}{4\pi i}\int_{\partial D_r}\frac{\Delta_n^*(z)}{\Delta_n(z)}\,dz=0.$$

EXAMPLE 3.2. For illustration, we consider the case $w(\tau)=1$ and $\Delta_n(\tau)=\sin(\frac{1}{2}n\tau+\theta)$ with arbitrary real θ. In this case, $t_j=(2/n)(j\pi+[-\theta]_\pi)$ $(j=0, ..., n-1)$ and

$$\Delta_n^*(z)=\begin{cases}2\pi e^{i((1/2)\,nz+\theta)}, & \text{if}\quad \operatorname{Im} z>0,\\ 2\pi\cos(\frac{1}{2}n\tau+\theta), & \text{if}\quad z=\tau, \operatorname{Im} z=0,\\ 2\pi e^{-i((1/2)\,nz+\theta),} & \text{if}\quad \operatorname{Im} z<0,\end{cases}\tag{3.25}$$

$$\mathbf{Q}_n^{\Delta D}f=\frac{2\pi}{n}\sum_{j=0}^{n-1}f(t_j),\tag{3.26}$$

and if $f\in AP(D_r)$, from (3.23),

$$|\mathbf{R}_n^{\Delta D}f|\leqslant 2\pi(\coth\tfrac{1}{2}nr-1)\,\|f\|_r=O(e^{-nr})\qquad(\text{as } n\to\infty).\tag{3.27}$$

We get also the following corollary noting

$$\|\Delta_n^*\|\leqslant\coth(\tfrac{1}{2}r)\,\mathbf{D}1.\tag{3.28}$$

COROLLARY 3.2. *If $f\in AP(D_r)$, then*

$$|\mathbf{R}_n^{\Delta D}f|\leqslant\coth(\tfrac{1}{2}r)\,\mathbf{D}1\,\sinh^{-n}(\tfrac{1}{2}r)\,\|f\|_r.\tag{3.29}$$

In particular, when $r>2\ln(1+\sqrt{2})$, then $|\mathbf{R}_n^{\Delta D}f|\to 0$ as $n\to\infty$.

4. QUADRATURE FORMULAS FOR SINGULAR INTEGRALS

In this section, we discuss quadrature formulas for (1.1). We need further some properties of the associated function Δ_n^*.

LEMMA 4.1. *When $t\in[0,2\pi)$ and $t\neq t_j$ $(j=1, ..., n)$,*

$$\frac{\Delta_n^*(t)}{\Delta_n(t)}=(\mathbf{H}1)(t)-\sum_{j=1}^{n}H_j\cot\frac{t_j-t}{2}.$$

Proof. When $n=2m-1$, by (2.14), Lemma 2.3, and Theorem 3.1, we have

$$(\tilde{\mathbf{H}}\Delta_n)(\tau)=\int_0^{2\pi} w(\tau)\Lambda_n(\tau,t)\,d\tau+\Delta_n(t)\int_0^{2\pi} w(\tau)\cot\frac{\tau-t}{2}\,d\tau$$

$$=-\sum_{j=1}^{n} H_j\Delta_n(t)\cot\frac{t_j-t}{2}+\Delta_n(t)(\mathbf{H}1)(t).$$

This is just what we want to prove. The proof for the case of $n=2m$ is similar.

We have pointed out before that Δ_n^* is continuous at $z=t_j$. Now we further prove that it has the following sectional derivatives at $z=t_j$.

LEMMA 4.2. *Let*

$$\Delta_{n,j}(\tau)=\Delta_n(\tau)\csc\frac{\tau-t_j}{2}, \tag{4.1}$$

then

$$(\Delta_n^*|_{\mathscr{R}})'(t_j)=:\lim_{z\to 0,\, z\in\mathscr{R}}\frac{\Delta_n^*(z)-\Delta_n^*(t_j)}{z-t_j}$$
$$=\begin{cases}\frac{1}{2}(\mathbf{H}\Delta_{n,j})(t_j), & \text{if } n=2m-1,\\ \frac{1}{2}(\tilde{\mathbf{H}}\Delta_{n,j})(t_j), & \text{if } n=2m;\end{cases} \tag{4.2}$$

$$(\Delta_n^*|_{D_r^+})'(t_j)=:\lim_{z\to 0,\, z\in D_r^+}\frac{\Delta_n^*(z)-\Delta_n^*(t_j)}{z-t_j}$$
$$=\begin{cases}\frac{1}{2}(\mathbf{H}^+\Delta_{n,j})(t_j), & \text{if } n=2m-1,\\ \frac{1}{2}(\tilde{\mathbf{H}}^+\Delta_{n,j})(t_j), & \text{if } n=2m;\end{cases} \tag{4.3}$$

$$(\Delta_n^*|_{D_r^-})'(t_j)=:\lim_{z\to 0,\, z\in D_r^-}\frac{\Delta_n^*(z)-\Delta_n^*(t_j)}{z-t_j}$$
$$=\begin{cases}\frac{1}{2}(\mathbf{H}^-\Delta_{n,j})(t_j), & \text{if } n=2m-1,\\ \frac{1}{2}(\tilde{\mathbf{H}}^-\Delta_{n,j})(t_j), & \text{if } n=2m.\end{cases} \tag{4.4}$$

Proof. We only prove the case $n=2m-1$. Noting

$$\frac{\Delta_n^*(z)-\Delta_n^*(t_j)}{z-t_j}=\frac{\sin(1/2)(z-t_j)}{z-t_j}(\mathbf{H}\Delta_{n,j})(z)+\frac{\cos(1/2)(z-t_j)-1}{z-t_j}\mathbf{D}\Delta_{n,j},$$

by Remark 3.2, we obtain (4.2)–(4.4).

Remark 4.1. We know that (4.2) is the derivative of the restricted function $\Delta_n^*|_{\mathscr{R}}$ of Δ_n^* on $\mathscr{R}$, which is often used. For simplicity, we agree that the derivative $(\Delta_n^*|_{\mathscr{R}})'(t_j)$ is always denoted by $(\Delta_n^*)'(t_j)$.

Now we use the method of separation of singularities to derive quadrature formulas for the singular integral (1.1).

$$\begin{aligned}(\mathbf{H}f)(t) &= \int_0^{2\pi} w(\tau)[f(\tau)-f(t)]\cot\frac{\tau-t}{2}\,d\tau + f(t)(\mathbf{H}1)(t)\\ &\approx \sum_{j=1}^n H_j[f(t_j)-f(t)]\cot\frac{t_j-t}{2} + f(t)(\mathbf{H}1)(t)\\ &= \sum_{j=1}^n H_j f(t_j)\cot\frac{t_j-t}{2} + \frac{\Delta_n^*(t)}{\Delta_n(t)}\,f(t).\end{aligned}$$

In the last equality we have used (2.12), Example 3.1, and Lemma 4.1.

For

$$(\mathbf{Q}_n^{\Delta H}f)(t) = \sum_{j=1}^n H_j\cot\frac{t_j-t}{2}\,f(t_j) + \frac{\Delta_n^*(t)}{\Delta_n(t)}\,f(t), \tag{4.5}$$

we obtain the quadrature formula of (1.1):

$$(\mathbf{H}f)(t) \approx (\mathbf{Q}_n^{\Delta H}f)(t). \tag{4.6}$$

In the above quadrature sum, when t coincides with some t_j, we must understand $(\mathbf{Q}_n^{\Delta D}f)(t)$ as its limit value, namely,

$$(\mathbf{Q}_n^{\Delta H}f)(t_j) = \sum_{r=1,\, r\neq j}^n H_r\cot\frac{t_r-t_j}{2}\,f(t_r) + K_j f(t_j) + 2H_j f'(t_j), \tag{4.7}$$

where

$$K_j = \frac{(\Delta_n^*)'(t_j)}{\Delta_n'(t_j)} - H_j\,\frac{\Delta_n''(t_j)}{\Delta_n'(t_j)}, \tag{4.8}$$

with $(\Delta_n^*)'(t_j)$ given in (4.2) of Lemma 4.2 (recalling the agreement in Remark 4.1). In fact, by (3.19) and (4.1)

$$\begin{aligned}K_j &= \lim_{t\to t_j}\left[H_j\cot\frac{t_j-t}{2} + \frac{\Delta_n^*(t)}{\Delta_n(t)}\right] = 2\left[\frac{\Delta_n^*(t)}{\Delta_{n,j}(t)}\right]'_{t=t_j}\\ &= \frac{(\Delta_n^*)'(t_j)}{\Delta_n'(t_j)} - H_j\,\frac{\Delta_n''(t_j)}{\Delta_n'(t_j)}.\end{aligned}$$

Summarizing the above results, we have the following theorem.

THEOREM 4.1. *If $f \in C'_{2\pi}$, then*

$$(\mathbf{H}f)(t) = (\mathbf{Q}_n^{\Delta H} f)(t) + (\mathbf{R}_n^{\Delta H} f)(t), \tag{4.9}$$

where

$$(\mathbf{Q}_n^{\Delta H} f)(t) = \begin{cases} \sum_{j=1}^{n} H_j f(t_j) \cot \dfrac{t_j - t}{2} + \dfrac{\Delta_n^*(t)}{\Delta_n(t)} f(t), \\ \quad \text{if} \quad t \neq t_j, \\ \sum_{r=1,\, r \neq j}^{n} H_r \cot \dfrac{t_r - t_j}{2} f(t_r) + K_j f(t_j) + 2H_j f'(t_j), \\ \quad \text{if} \quad t = t_j, \end{cases} \tag{4.10}$$

where Δ_n^*, H_j, K_j are given by (3.18), (3.19), (4.8), *respectively, and*

$$(\mathbf{R}_n^{\Delta H} f)(t) = \mathbf{R}_n^{\Delta D} F_t = \int_0^{2\pi} w(\tau)(\delta_n^{\Delta} F_t)(\tau)\, d\tau. \tag{4.11}$$

We treat the concepts of the trigonometric precision and the type of the singular quadrature formula (4.9) similarly to those of (3.2) in Definition 3.1 and Definition 3.3.

COROLLARY 4.1. *For* (4.6), $\mathrm{pr}(\mathbf{Q}_n^{\Delta H}) \geqslant [\frac{1}{2}(n-1)]$ *and* $\mathrm{ty}\{\mathbf{Q}_n^{\Delta H}\} \supseteq H_n^T(\phi)$ *when $n = 2m$, with ϕ being just the characteristic number of Δ_n.*

Remark 4.2. We point out that (4.6) need not be of interpolation type. In fact, if it is of interpolation type, then it is necessary that $\mathrm{pr}(\mathbf{Q}_n^{\Delta H}) \geqslant [\frac{1}{2}n]$ because the number of its nodes $\{t_j, t\}$ is $n+1$ instead of n, but only $\mathrm{pr}(\mathbf{Q}_n^{\Delta H}) \geqslant [\frac{1}{2}(n-1)]$ from Corollary 4.1. So, the quadrature formula (4.6) derived via the method of separation of singularities is called the quadrature formula of quasi-interpolation type. This fact is very surprising and does not happen in the corresponding discussion of singular integrals with Cauchy kernel. The quadrature formula of interpolation type for (1.1) will be discussed in another paper.

THEOREM 4.2. *If $f \in AP(D_r)$, then the remainder of* (4.9)

$$(\mathbf{R}_n^{\Delta H} f)(t) = -\frac{1}{4\pi i} \int_{\partial D_r} f(z) \frac{\Delta_n^*(z)}{\Delta_n(z)} \cot \frac{z-t}{2}\, dz, \tag{4.12}$$

$$(\mathbf{R}_n^{\Delta H} f)(t) = -\frac{1}{2\pi} \operatorname{Re} \left\{ i \int_{ir}^{2\pi + ir} f(z) \frac{\Delta_n^*(z)}{\Delta_n(z)} \cot \frac{z-t}{2}\, dz \right\}, \tag{4.13}$$

and

$$\|\mathbf{R}_n^{\Delta H} f\| \leqslant \coth\left(\frac{r}{2}\right) \|f\|_r \left\|\frac{\Delta_n^*}{\Delta_n}\right\|_r. \tag{4.14}$$

Remark 4.3. In the first place, we need to explain in detail the integral in (4.12): when $\Delta_n(0)=0$ and $t \neq 0(2\pi)$ it may be understood as the Cauchy principal value integral by Remark 3.3; when $\Delta_n(0) \neq 0$ and $t=0(2\pi)$ it may be understood as the (one-sided) Cauchy principal value integral because both $(\Delta_n^*)^+(0)$ and $(\Delta_n^*)^-(0)$ exist by Remark 3.2; when both $\Delta_n(0)=0$ and $t=0(2\pi)$ it may be understood as the (one-sided) higher order singular integral because both $(\Delta_n^*|_{\partial D_r^+})'(0)$ and $(\Delta_n^*|_{\partial D_r^-})'(0)$ exist by Lemma 4.2. The definition of the (one-sided) singular integral may be seen in [8, 9].

Under this interpretation, from (3.21) and (4.11), we get

$$\begin{aligned}(\mathbf{R}_n^{\Delta H} f)(t) &= -\frac{1}{4\pi i}\int_{\partial D_r} [f(z)-f(t)] \cot\frac{z-t}{2} \frac{\Delta_n^*(z)}{\Delta_n(z)}\, dz \\ &= -\frac{1}{4\pi i}\int_{\partial D_r} f(z) \frac{\Delta_n^*(z)}{\Delta_n(z)} \cot\frac{z-t}{2}\, dz \\ &\quad +\frac{f(t)}{4\pi i}\int_{\partial D_r} \frac{\Delta_n^*(z)}{\Delta_n(z)} \cot\frac{z-t}{2}\, dz.\end{aligned}$$

Thus, we only need to prove the following lemma which possesses independent significance.

LEMMA 4.3.

$$\frac{1}{4\pi i}\int_{\partial D_r} \frac{\Delta_n^*(z)}{\Delta_n(z)} \cot\frac{z-t}{2}\, dz = 0. \tag{4.15}$$

Proof. First assume that $t \neq t_j$. By Remark 3.2 and the extended residue theorem [5],

$$\frac{1}{2\pi i}\int_{\partial D_r^+} \frac{(\Delta_n^*)^+(z)}{\Delta_n(z)} \cot\frac{z-t}{2}\, dz = \mathrm{sp}(t)\,\mathrm{res}(t) + \sum_{j=1}^{n} \mathrm{sp}(t_j)\,\mathrm{res}(t_j), \tag{4.16}$$

where $\mathrm{sp}(x)$ denotes the span at x with respect to D_r^+ and $\mathrm{res}(x)$ the residue of the integrand $[(\Delta_n^*)^+(z)/\Delta_n(z)]\cot\frac{1}{2}(z-t)$ at x. Noting $\mathrm{sp}(t)=\frac{1}{2}$, $\mathrm{sp}(t_j)=\frac{1}{2}$ since D_r^+ is a rectangular domain, we get

$$\frac{1}{2\pi i}\int_{\partial D_r^+}\frac{(\Delta_n^*)^+(z)}{\Delta_n(z)}\cot\frac{z-t}{2}\,dz$$
$$=\frac{(\Delta_n^*)^+(t)}{\Delta_n(t)}+\sum_{j=1}^{n}\frac{(\Delta_n^*)^+(t_j)}{2\Delta_n'(t_j)}\cot\frac{t_j-t}{2}. \tag{4.17}$$

It must be pointed out that one of t and t_j is possibly 0, say $t_1=0$, in this instance $\mathrm{sp}(t_1)=1/4$, but since the periodicity of the integrand $[(\Delta_n^*)^+(z)/\Delta_n(z)]\cot\frac{1}{2}(z-t)$, 2π is also a singular point, hence, in the sum of the right-hand side of (4.16) we must replace the term $\mathrm{sp}(t_1)\,\mathrm{res}(t_1)$ by the sum $\mathrm{sp}(0)\,\mathrm{res}(0)+\mathrm{sp}(2\pi)\,\mathrm{res}(2\pi)$ which still is equal to $\frac{1}{2}\mathrm{res}(t_1)$. Therefore (4.17) still holds. In the same way, we have

$$\frac{1}{2\pi i}\int_{\partial D_r^-}\frac{(\Delta_n^*)^-(z)}{\Delta_n(z)}\cot\frac{z-t}{2}\,dz$$
$$=\frac{(\Delta_n^*)^-(t)}{\Delta_n(t)}+\sum_{j=1}^{n}\frac{(\Delta_n^*)^-(t_j)}{2\Delta_n'(t_j)}\cot\frac{t_j-t}{2}. \tag{4.18}$$

From (3.14)–(3.15) and (3.20), we know

$$\begin{cases}(\Delta_n^*)^+(\tau)+(\Delta_n^*)^-(\tau)=2\Delta_n^*(\tau),\\ (\Delta_n^*)^+(\tau)-(\Delta_n^*)^-(\tau)=4\pi i w(\tau)\,\Delta_n(\tau),\end{cases}\qquad \tau\in\mathscr{R}. \tag{4.19}$$

Therefore, by Lemma 4.1 and (4.17)–(4.19), we obtain

$$\frac{1}{4\pi i}\int_{\partial D_r}\frac{\Delta_n^*(z)}{\Delta_n(z)}\cot\frac{z-t}{2}\,dz$$
$$=\frac{1}{4\pi i}\int_{\partial D_r^+}\frac{(\Delta_n^*)^+(z)}{\Delta_n(z)}\cot\frac{z-t}{2}\,dz+\frac{1}{4\pi i}\int_{\partial D_r^-}\frac{(\Delta_n^*)^-(z)}{\Delta_n(z)}\cot\frac{z-t}{2}\,dz$$
$$-\frac{1}{4\pi i}\int_0^{2\pi}\frac{(\Delta_n^*)^+(\tau)-(\Delta_n^*)^-(\tau)}{\Delta_n(\tau)}\cot\frac{\tau-t}{2}\,d\tau$$
$$=\frac{\Delta_n^*(t)}{\Delta_n(t)}+\sum_{j=1}^{n}H_j\cot\frac{t_j-t}{2}-(\mathbf{H}1)(t)=0.$$

Applying the technique used in (3.24), we also get

$$\begin{aligned}
&\frac{1}{4\pi i}\int_{\partial D_r}\frac{\Delta_n^*(z)}{\Delta_n(z)}\cot\frac{z-t_j}{2}\,dz\\
&=\frac{1}{4\pi i}\left\{\int_{-ir}^{2\pi-ir}-\int_{ir}^{2\pi+ir}\right\}\frac{\Delta_n^*(z)}{\Delta_n(z)}\cot\frac{z-t_j}{2}\,dz\\
&=\lim_{t\to t_j}\frac{1}{4\pi i}\left\{\int_{-ir}^{2\pi-ir}-\int_{ir}^{2\pi+ir}\right\}\frac{\Delta_n^*(z)}{\Delta_n(z)}\cot\frac{z-t}{2}\,dz\\
&=\lim_{t\to t_j}\frac{1}{4\pi i}\int_{\partial D_r}\frac{\Delta_n^*(z)}{\Delta_n(z)}\cot\frac{z-t}{2}\,dz=0.
\end{aligned}$$

This completes the proof of this lemma and hence Theorem 4.2.

Corollary 4.2 *When*

$$f\in\begin{cases} H_{m-1}^T, & \textit{if}\quad n=2m-1,\\ H_m^T(\phi), & \textit{if}\quad n=2m,\end{cases}$$

where ϕ is the characteristic number of Δ_n, then

$$\frac{1}{4\pi i}\int_{\partial D_r} f(z)\,\frac{\Delta_n^*(z)}{\Delta_n(z)}\cot\frac{z-t}{2}\,dz=0.$$

Example 4.1. In the case stated in Example 3.2, we have

$$(\mathbf{Q}_n^{\Delta H}f)(t)=\begin{cases}\dfrac{2\pi}{n}\displaystyle\sum_{j=0}^{n-1} f(t_j)\cot\frac{t_j-t}{2}+2\pi\cot\left(\frac{1}{2}nt+\theta\right)f(t),\\ \quad\text{if}\quad t\neq t_j,\\ \dfrac{2\pi}{n}\displaystyle\sum_{j=0,\,j\neq r}^{n-1} f(t_j)\cot\frac{t_j-t_r}{2}+\frac{4\pi}{n}f'(t_r),\\ \quad\text{if}\quad t=t_r,\end{cases}\tag{4.20}$$

where $t_j=(2/n)(j\pi+[-\theta]_\pi)$ and if $f\in AP(D_r)$, from (4.14)

$$\|\mathbf{R}_n^{\Delta H}f\|\leqslant 2\pi\coth\tfrac{1}{2}r[\coth\tfrac{1}{2}nr-1]\;\|f\|_r.\tag{4.21}$$

It follows that $\|\mathbf{R}_n^{\Delta H}f\|=O(e^{-nr})$ as $n\to\infty$. The result with $\theta=0$ is just the same as that obtained by Chawla, Ramakrishnan, and Ioakimidis [10, 11]. It should be noted that the first two authors obtained this quadrature formula only for the case n is even and Ioakimidis did not give any estimate of the remainder.

COROLLARY 4.3. *If $f \in AP(D_r)$, then, from* (4.14)

$$\|\mathbf{R}_n^{\Delta H} f\| \leqslant \coth\left(\tfrac{1}{2}r\right) \mathbf{D}1\left(\sinh \tfrac{1}{2}r\right)^{-n} \|f\|_r. \tag{4.22}$$

In particular, when $r > 2\ln(1+\sqrt{2})$ *then* $\|\mathbf{R}_n^{\Delta H} f\| \to 0$ *as* $n \to \infty$.

We have seen that the singular quadrature operator $\mathbf{Q}_n^{\Delta H}$ must be defined on $C'_{2\pi}$, or rather on $C'_{2\pi}(\Delta_n)$ which denotes the class of all functions belonging to $C_{2\pi}$ and with derivatives at t_j ($j=1, ..., n$). This is not convenient in applications. For this reason, we introduce an interpolation operator as

$$\mathbf{L}_n^{\Delta} : C_{2\pi} \to C'_{2\pi}(\Delta_n) \tag{4.23}$$

with the interpolation property

$$(\mathbf{L}_n^{\Delta} f)(t_j) = f(t_j) \qquad (j=1, ..., n). \tag{4.24}$$

The trigonometric precision of $\mathbf{L}_n^{\Delta}$ is defined by

$$\operatorname{pr}(\mathbf{L}_n^{\Delta}) = \max\{j, H_j^T \subseteq \ker(\mathbf{I} - \mathbf{L}_n^{\Delta})\}. \tag{4.25}$$

We agree that $\operatorname{pr}(\mathbf{L}_n^{\Delta}) = -\infty$ if $1 \notin \ker(\mathbf{I} - \mathbf{L}_n^{\Delta})$ and $\operatorname{pr}(\mathbf{L}_n^{\Delta}) = +\infty$ if $H_j^T \subseteq \ker(\mathbf{I} - \mathbf{L}_n^{\Delta})$ for any integer j.

There are many such operators, for example, $\mathbf{T}_n^{\Delta}$ given in (2.3). Replacing f by $\mathbf{L}_n^{\Delta} f$ in the quadrature operator (4.10), we obtain $\mathbf{Q}_n^{\Delta H} \mathbf{L}_n^{\Delta} f$, denoted as $(\mathbf{QL})_n^{\Delta H}$, i.e.,

$$((\mathbf{QL})_n^{\Delta H} f)(t) = \begin{cases} \displaystyle\sum_{j=1}^{n} H_j \cot\frac{t_j - t}{2} f(t_j) + \frac{\Delta_n^*(t)}{\Delta_n(t)} (\mathbf{L}_n^{\Delta} f)(t), \\ \qquad \text{if} \quad t \neq t_j, \\ \displaystyle\sum_{r=1,\, r\neq j}^{n} H_r \cot\frac{t_r - t_j}{2} f(t_r) + K_j f(t_j) + 2H_j (\mathbf{L}_n^{\Delta} f)'(t_j), \\ \qquad \text{if} \quad t = t_j, \end{cases} \tag{4.26}$$

where K_j is as in (4.8).

Obviously, we have

$$\operatorname{pr}((\mathbf{QL})_n^{\Delta H}) \geqslant \min\{\operatorname{pr}(\mathbf{Q}_n^{\Delta H}), \operatorname{pr}(\mathbf{L}_n^{\Delta})\}. \tag{4.27}$$

Taking $\mathbf{L}_n^{\Delta}=\mathbf{I}$ on $C'_{2\pi}$, we reobtain the previous result. For another example, taking $\mathbf{L}_n^{\Delta}=\mathbf{T}_n^{\Delta}$ where $\mathbf{T}_n^{\Delta}$ is given as in (2.10), we have

$$((\mathbf{QT})_n^{\Delta H} f)(t)=\sum_{j=1}^{n} A_j(t)\, f(t_j), \tag{4.28}$$

where

$$A_j(t)=\begin{cases} \dfrac{\Delta_n^*(t,t_j)}{2\Delta_n'(t_j)}, & \text{if} \quad t\neq t_j, \\ \dfrac{(\Delta_n^*)'\,(t_j)}{\Delta_n'(t_j)}, & \text{if} \quad t=t_j, \end{cases} \tag{4.29}$$

in which $(\Delta_n^*)'\,(t_j)$ is (4.2) in Lemma 4.2 and

$$\Delta_n^*(\tau,t)=\begin{cases} \left[\Delta_n^*(\tau)-\Delta_n^*(t)\cos\dfrac{\tau-t}{2}\right]\csc\dfrac{\tau-t}{2}, & \text{if} \quad n=2m-1, \\ [\Delta_n^*(\tau)-\Delta_n^*(t)]\cot\dfrac{\tau-t}{2}, & \text{if} \quad n=2m. \end{cases} \tag{4.30}$$

Obviously, by Lemma 2.1 and Corollary 4.1,

$$\operatorname{pr}((\mathbf{QT})_n^{\Delta H})\geqslant[\tfrac{1}{2}(n-1)]. \tag{4.31}$$

Hence, (4.28) is of trigonometric interpolation type (noting the number of its nodes is n). This generalizes the result in [12].

EXAMPLE 4.2. In the case given in Example 4.1, for simplicity, with $\theta=0$, (4.29) becomes

$$A_j(t)=\begin{cases} \dfrac{2\pi}{n}\left[\cos\dfrac{n(t-t_j)}{2}-\cos\dfrac{t-t_j}{2}\right]\csc\dfrac{t-t_j}{2}, & \text{if } n \text{ is odd}, \\ (-1)^j\dfrac{2\pi}{n}\left[\cos\dfrac{n}{2}t-\cos\dfrac{n}{2}t_j\right]\cot\dfrac{t-t_j}{2}, & \text{if } n \text{ is even}, \end{cases} \tag{4.32}$$

and $A_j(t_j)=0$. This is just the result given in [13] by S. Krenk.

5. CONVERGENCE

In this section, we discuss the convergence of the quadrature formulas established above. Given a sequence $\{\Delta_n\}$, we obtain a sequence of quadrature formulas of interpolation type $\mathbf{Q}_n^{\Delta D}$. If $\lim_{n\to\infty}\mathbf{R}_n^{\Delta D} f=0$, we say it is convergent for f. For example, (3.26) is convergent for any $f\in AP(D_r)$.

In applications, we need to consider more general functions and weights. We assume that the spaces considered are equipped with Chebyshev's norm, namely, the maximum of the absolute values of a function. We now rewrite the coefficients H_j of $\mathbf{Q}_n^{\Delta D}$ as $H_{n,j}$ to avoid confusion.

Obviously, the norm of the quadrature operator $\mathbf{Q}_n^{\Delta D}$ is

$$\|\mathbf{Q}_n^{\Delta D}\| = \sum_{j=1}^{n} |H_{n,j}|, \tag{5.1}$$

and the norm of the integral operator $\mathbf{D}$ is

$$\|\mathbf{D}\| = \mathbf{D}1. \tag{5.2}$$

THEOREM 5.1. *If* $\|\mathbf{Q}_n^{\Delta D}\| = o(n^\mu)$ $(0 < \mu \leqslant 1)$, *then the sequence* $\{\mathbf{Q}_n^{\Delta D}\}$ *is convergent for* $f \in H_{2\pi}^\mu$ *(the class of functions with Hölder index* μ *in* $H_{2\pi}$*). More precisely,* $|\mathbf{R}_n^{\Delta D} f| \leqslant 12\ (\|\mathbf{D}\| + \|\mathbf{Q}_n^{\Delta D}\|)\ \omega(f, 1/m)$ *for* $f \in C_{2\pi}$, *where* $\omega(f, \cdot)$ *is the modulus of continuity of* f *and* $m = [\frac{1}{2}(n-1)]$.

Proof. Denoting the best approximating trigonometric polynomial of degree not greater than m of f by J_m, then $\|f - J_m\| \leqslant 12\omega(f, 1/m)$ by Jackson's theorem. Noting that $\mathbf{Q}_n^{\Delta D}$ is of interpolation type, we get $\mathbf{R}_n^{\Delta D} f = (\mathbf{D} - \mathbf{Q}_n^{\Delta D})(f - J_m)$, therefore $|\mathbf{R}_n^{\Delta D} f| \leqslant 12(\|\mathbf{D}\| + \|\mathbf{Q}_n^{\Delta D}\|)\ \omega(f, 1/m)$. Furthermore, $\lim_{n\to\infty} \mathbf{R}_n^{\Delta D} f = 0$ for $f \in H_{2\pi}^\mu$ by noting that $\|\mathbf{Q}_n^{\Delta D}\| = o(n^\mu)$.

Remark 5.1. If $\{\mathbf{Q}_n^{\Delta D}\}$ is a sequence of non-negative operators, i.e., $H_{n,j} \geqslant 0$ $(j = 1, ..., n)$, for example, the sequence of the quadrature formulas with the highest trigonometric precision established in [1] ((3.26) is just a special example), then $\|\mathbf{Q}_n^{\Delta D}\| = \sum_{j=1}^n H_{n,j} = \mathbf{D}1 = \|\mathbf{D}\|$ holds and the sequence is convergent for any $f \in C_{2\pi}$.

Now we discuss the convergence of the sequence $\{\mathbf{Q}_n^{\Delta H}\}$.

LEMMA 5.1. *If* $f \in C'_{2\pi}$, *then* $\omega(F, h) \leqslant C\omega(f', h)$, *where* F *is given in* (2.12) *and* C *is a constant independent of* h.

Proof. Assume $h \leqslant \pi/4$ first. When $\max\{|\Delta\tau|, |\Delta t|\} \leqslant h$, we come to estimate the difference $\Delta F = F(\tau + \Delta\tau, t + \Delta t) - F(\tau, t)$.

When $0 \leqslant \tau - t \leqslant \pi$, we set

$$F^*(\tau, t) = \begin{cases} \dfrac{f(\tau) - f(t)}{\tau - t}, & \text{if} \quad \tau \neq t, \\ f'(t), & \text{if} \quad \tau = t, \end{cases} \tag{5.3}$$

$$\phi(x) = x \cot x, \qquad -\pi < x < \pi. \tag{5.4}$$

It is easy to check

$$\max_{|x| \leqslant (1/2)\pi} |\phi(x)| = 1, \qquad \max_{|x| \leqslant (3/4)\pi} |\phi'(x)| = |\phi'(\tfrac{3}{4}\pi)| < 6 \tag{5.5}$$

and

$$F(\tau, t) = 2F^*(\tau, t)\, \phi\left(\frac{\tau - t}{2}\right). \tag{5.6}$$

Now we have, by Lemma 2 in [6],

$$|\Delta F| \leqslant 2(|\Delta F^*| + 6\, \|f'\|\, h) \leqslant 2(\omega(f', h) + 6\, \|f'\|\, h). \tag{5.7}$$

When $\pi \leqslant \tau - t \leqslant 2\pi$, setting

$$F^*(\tau, t) = \begin{cases} \dfrac{f(\tau) - f(2\pi + t)}{\tau - (2\pi + t)}, & \text{if} \quad \tau \not\equiv t \pmod{2\pi}, \\ f'(t), & \text{if} \quad \tau \equiv t \pmod{2\pi}, \end{cases} \tag{58}$$

we again have

$$F(\tau, t) = 2F^*(\tau, t)\, \phi\left(\pi - \frac{\tau - t}{2}\right). \tag{5.9}$$

Now working in a way analogous to that used above, we can show that (5.7) still holds. From the property of the modulus of continuity,

$$\frac{1}{2}\, \frac{\omega(f', \pi/4)}{\pi/4} \leqslant \frac{\omega(f', h)}{h}. \tag{5.10}$$

We treat $f' \neq \text{const}$ (otherwise, $f = \text{const}$ since $f \in C_{2\pi}$ and hence $F \equiv 0$, which is the trivial case), therefore, by noting (5.7) and (5.10),

$$\omega(F, h) \leqslant C\omega(f', h) \qquad (h \leqslant \pi/4), \tag{5.11}$$

where

$$C = 2\left[1 + \frac{3\pi\, \|f'\|}{\omega(f', \pi/4)}\right]. \tag{5.12}$$

If $h > \pi/4$, then

$$\omega(F, h) \leqslant 2\, \|F\| \leqslant \frac{4\, \|f'\|}{\omega(f', \pi/4)}\, \omega(f', h). \tag{5.13}$$

Therefore (5.11) is true for any h.

By this lemma, (4.11), and Theorem 5.1, we have the following theorem.

THEOREM 5.2. *If* $\|\mathbf{Q}_n^{\Delta D}\| = o(n^\mu)$ $(0 < \mu \leqslant 1)$, *then the sequence* $\{\mathbf{Q}_n^{\Delta H}\}$ *is convergent for* $f \in H_{2\pi}^{1,\mu}$ *(the class of all functions* $f' \in H_{2\pi}^\mu$*), i.e.,* $\lim_{n\to\infty} \|\mathbf{R}_n^{\Delta H} f\| = 0$. *More precisely,* $\|\mathbf{R}_n^{\Delta H} f\| \leqslant 12C(\|\mathbf{D}\| + \|\mathbf{Q}_n^{\Delta D}\|)\,\omega(f', 1/m)$ *for* $f' \in C_{2\pi}$, *where* $m = [\frac{1}{2}(n-1)]$ *and* C *is* 0 *if* $f = \mathrm{const}$ *or as in* (5.12) *if not.*

Remark 5.2. If $\{\mathbf{Q}_n^{\Delta D}\}$ is a sequence of non-negative operators, for example, those established in [1] or (3.26), then $\{\mathbf{Q}_n^{\Delta H}\}$ is convergent for any $f \in C'_{2\pi}$.

The following lemma will be also used in the sequel (cf. [14, Hilfssatz 2, Sect. 2, Kapitel 2]).

LEMMA 5.2. *Suppose that* $f \in C_{2\pi}$. *If there exists a sequence of trigonometric polynomials* $T_n \in H_n^T (j = 1, ...,)$ *such that*

$$\|f - T_n\| \leqslant \frac{A_1}{n^\alpha} \qquad (A_1 = \mathrm{const},\ 0 < \alpha \leqslant 1), \tag{5.14}$$

then

$$M[f_n, \beta] \leqslant \frac{A_2}{n^{\alpha-\beta}}, \tag{5.15}$$

where $f_n(t) = f(t) - T_n(t)$, $0 < \beta \leqslant \alpha$ *if* $0 < \alpha < 1$ *or* $0 < \beta < 1$ *if* $\alpha = 1$, A_2 *is a constant only depending on* α *and* β, *and*

$$M[f, \beta] = \sup_{|\tau - t| > 0} \frac{|f(\tau) - f(t)|}{|\tau - t|^\beta}. \tag{5.16}$$

THEOREM 5.3. *If* $\|(\mathbf{QL})_n^{\Delta H}\| = o(n^\mu)$ $(0 < \mu \leqslant 1)$ *and* $\mathrm{pr}((\mathbf{QL})_n^{\Delta H}) \geqslant m = [\frac{1}{2}(n-1)]$, *then the sequence* $\{(\mathbf{QL})_n^{\Delta H}\}$ *is convergent for* $f \in H_{2\pi}^\mu$, *i.e.,* $\lim_{n\to\infty} \|(\mathbf{RL})_n^{\Delta H} f\| = 0$, *where* $(\mathbf{RL})_n^{\Delta H} f = \mathbf{H}f - (\mathbf{QL})_n^{\Delta H} f$.

Proof. Denoting the best approximate polynomial of degree not greater than m of f by J_m and $f_m(\tau) = f(\tau) - J_m(\tau)$, then, by the Jackson theorem, $\|f_m\| \leqslant 12\omega(f, 1/m)$, and hence, for $0 < \varepsilon < \mu$,

$$\begin{aligned} |(\mathbf{H}f_m)(t)| &= \left| f_m(t)(\mathbf{H}1)(t) + \int_0^{2\pi} w(\tau)[f_m(\tau) - f_m(t)] \cot\frac{\tau - t}{2}\, d\tau \right| \\ &\leqslant 12\, \|\mathbf{H}1\|\, \omega\left(f, \frac{1}{m}\right) \\ &\quad + \left| \int_{-\pi+t}^{\pi+t} w(\tau) \frac{f_m(\tau) - f_m(t)}{\tau - t} \left[(\tau - t) \cot\frac{\tau - t}{2} \right] d\tau \right| \end{aligned}$$

$$\leqslant 12 \,\|\mathbf{H1}\|\, \omega\left(f, \frac{1}{m}\right) + \frac{2A_2 \,\|w\|}{m^{\mu-\varepsilon}} \int_{-\pi+t}^{\pi+t} \frac{d\tau}{|\tau - t|^{1-\varepsilon}}$$

(by Lemma 5.2 and (5.5))

$$\leqslant \frac{12M[f, \mu] \,\|\mathbf{H1}\|}{m^{\mu}} + \frac{4\pi^{\varepsilon} A_2 \,\|w\|}{\varepsilon m^{\mu-\varepsilon}} \qquad \text{(by (5.16))},$$

where A_2 is a constant only depending on μ and ε. Again

$$\|(\mathbf{QL})_n^{\Delta H} f_m\| \leqslant \|(\mathbf{QL})_n^{\Delta H}\| \; \|f_m\| \leqslant \frac{12M[f, \mu] \,\|(\mathbf{QL})_n^{\Delta H}\|}{m^{\mu}},$$

and, by $\mathrm{pr}((\mathbf{QL})_n^{\Delta H}) \geqslant m$, $(\mathbf{RL})_n^{\Delta H} f = (\mathbf{RL})_n^{\Delta H} f_m$, we get $\lim_{n \to \infty} \|(\mathbf{RL})_n^{\Delta H} f\| = 0$.

COROLLARY 5.1. *If* $\|(\mathbf{QT})_n^{\Delta H}\| = o(n^{\mu})$ $(0 < \mu \leqslant 1)$, *then the sequence* $\{(\mathbf{QT})_n^{\Delta H}\}$ *is convergent for* $f \in H_{2\pi}^{\mu}$.

Remark 5.3. Because the functions considered are periodic, for more convenience, we regard $\mathscr{K}$ as the factor group $(-\infty, +\infty) \bmod 2\pi$ under addition and equipped with the value $|\tilde{x}|_{\mathscr{K}} = |x_0|$ for $\tilde{x} \in \mathscr{K}$, where $\tilde{x}$ is the congruence class of x, x_0 the congruence point lying in $[-\pi, \pi)$. In general, we use the symbol τ_t to denote the congruence point of τ, lying in $[-\pi + t, \pi + t)$. By this view, $(\mathscr{K}, |\cdot|_{\mathscr{K}})$ is a compact group, which will be very convenient in the following example.

EXAMPLE 5.1. For the $(\mathbf{QT})_n^{\Delta H}$ given in Example 4.2, we can prove that

$$\|(\mathbf{QT})_n^{\Delta H}\| \leqslant \left\| \sum_{j=0}^{n-1} |A_j(t)| \right\| \leqslant 4\pi^2 + 8 \ln n, \tag{5.17}$$

hence, (4.28) is convergent for $f \in H_{2\pi}^{\mu}$. In fact, we rearrange $\{(t_j)_t, j = 0, ..., n-1\}$ as $\{t_k^*, k = 1, ..., n\}$, according to

$$t_1^* < t_2^* < \cdots < t_n^*. \tag{5.18}$$

Obviously,

$$t_{k+1}^* - t_n^* = \frac{2\pi}{n}, \qquad \text{for} \quad k = 1, ..., n-1, \tag{5.19}$$

and there exists an integer v such that

$$t_v^* \leqslant t < t_{v+1}^*. \tag{5.20}$$

Writing

$$A_k^*(t) = A_j(t) \qquad \text{when} \quad t_k^* = (t_j)_t,$$

$$F_1(t) = \sum_{k=1}^{v-2} |A_k^*(t)|, \qquad F_2(t) = \sum_{k=v+3}^{n} |A_k^*(t)|,$$

with assumption $F_1(t) = 0$ if $v < 3$ and $f_2(t) = 0$ if $v > n-3$, we have, for odd n,

$$\begin{aligned} |A_k^*(t)| &= \frac{2\pi}{n} \left| 2 \sin \frac{n+1}{4} (t - t_k^*) \sin \frac{n-1}{4} (t - t_k^*) \csc \frac{t - t_k^*}{2} \right| \\ &\leqslant 2\pi \left| \frac{t - t_k^*}{2} \csc \frac{t - t_k^*}{2} \right| \qquad (\text{by } |\sin n\tau| \leqslant n\,|\tau|) \\ &\leqslant \pi^2. \end{aligned} \tag{5.21}$$

In the last inequality we have used

$$|\sin x| \geqslant \frac{2}{\pi} |x| \qquad \text{for} \quad |x| \leqslant \frac{\pi}{2}. \tag{5.22}$$

Then, by (5.19) and (5.22),

$$\begin{aligned} F_1(t) &= \sum_{k=1}^{v-2} \left| \int_{t_k^*}^{t_{k+1}^*} \left[\cos \frac{n(t - t_k^*)}{2} - \cos \frac{t - t_j^*}{2} \right] \csc \frac{t - t_k^*}{2}\, dx \right| \\ &\leqslant 2 \int_{-\pi+t}^{t_{v-1}^*} \csc \frac{t - x}{2}\, dx = -4 \ln \cot \frac{t - t_{v-1}^*}{4} \\ &\leqslant -4 \ln \sin \frac{\pi}{2n} \leqslant 4 \ln n. \end{aligned} \tag{5.23}$$

We obtain, in the same manner,

$$\|F_2\| \leqslant 4 \ln n. \tag{5.24}$$

For even n, by working in a similar way, (5.21) and (5.23)–(5.24) are again obtained. Finally, (5.17) results from (5.21) and (5.23)–(5.24).

Remark 5.4. It is rather difficult to estimate the order of $\|(\mathbf{QT})_n^{\Delta H}\|$ in the general case.

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